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# Brezis-Gallouet-Wainger type inequalities and blow-up criteria for Navier-Stokes equations in bounded domains

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## 1 Introduction

This note is a survey of our work [10]. Let  $\Omega$  be a 3-dimensional bounded domain with  $\partial\Omega \in C^\infty$ . The motion of a viscous incompressible fluid in  $\Omega$  is governed by the Navier-Stokes equations:

$$(N-S) \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0, & \operatorname{div} u = 0 & t \in (0, T), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, & u|_{t=0} = u_0, \end{cases}$$

where  $u = (u^1(x, t), u^2(x, t), u^3(x, t))$  and  $p = p(x, t)$  denote the velocity vector and the pressure, respectively, of the fluid at the point  $(x, t) \in \Omega \times (0, T)$  and  $u_0$  is the given initial velocity vector field. In this note, we consider Beale-Kato-Majda type blow-up criteria of classical solutions to (N-S).

Beale-Kato-Majda [1] and Kato-Ponce [7] showed that the  $L^\infty$ -norm of the vorticity  $\omega = \operatorname{rot} u$  controls the breakdown of smooth solutions to the Euler and Navier-Stokes equations if  $\Omega = \mathbb{R}^n$ . To be precise, if the smooth solution  $u$  in  $C([0, T]; W^{s,p}(\mathbb{R}^n))$  ( $s > n/p + 1$ ) breaks down at a finite time  $t = T$ , then  $\int_0^t \|\omega(\tau)\|_{L^\infty(\Omega)} d\tau \nearrow \infty$  as  $t \nearrow T$ . Chemin [4] and Kozono-Ogawa-Taniuchi [8] proved similar blow-up criteria with  $\|\omega\|_\infty$  replaced by  $\|u\|_{B^1_{\infty,\infty}}$  and  $\|\omega\|_{\dot{B}^0_{\infty,\infty}}$ . In the case where  $\Omega$  is a bounded domain, for the 3-D

Euler equations, Shirota-Yanagisawa [13] and Ferrari [6] proved the same result as Beale-Kato-Majda. See also Zajackowski [15]. Ogawa-Taniuchi [11] proved a similar blow-up criterion with  $\|\omega\|_{L^\infty(\Omega)}$  replaced by  $\|\omega\|_{bmo(\Omega)}$ . However, in [11], the blow-up criterion via  $\|\omega\|_{bmo(\Omega)}$  was proven only for 3-D Euler equations. In this note, we prove the same criterion for 3-D Navier-Stokes equations in bounded domains with smooth boundary.

In order to prove the above-mentioned results, the following Brezis-Gallouet-Wainger type inequalities play important roles:

$$(BGW)_\beta \quad \|f\|_{L^\infty} \leq C(1 + \|f\|_X \log^\beta(e + \|f\|_Y)).$$

Brezis-Gallouet-Wainger [2, 3] proved  $(BGW)_\beta$  with  $\beta = 1 - 1/p$ ,  $X = W^{n/p,p}(\mathbb{R}^n)$ ,  $Y = W^{n/q+\alpha,q}(\mathbb{R}^n) \subset \dot{C}^\alpha(\alpha > 0)$ . Engler [5] proved the same inequality for general domains  $\Omega$  without using the Fourier transform. We note that Ozawa [12] proved that  $\|f\|_{L^q(\mathbb{R}^n)} \leq C(p, n)q^{1-1/p}\|(-\Delta)^{n/2p}f\|_p^{1-p/q}\|f\|_p^{p/q}$  holds for all  $q \in [p, \infty)$  and that this estimate directly yields  $(BGW)_\beta$ . When  $\Omega$  is a bounded domain, in [11],  $(BGW)_\beta$  was proved for  $\beta = 1$ ,  $X = bmo(\Omega)$  and  $Y = \dot{C}^\alpha(\Omega)$  by using the method given in [12]. We note that in [1, 4, 5, 6, 7, 8, 9, 12, 13, 15] several inequalities of Brezis-Gallouet-Wainger type were established. Then, we have one question. What is the largest normed space  $X$  that satisfies  $(BGW)_\beta$  with  $Y = \dot{C}^\alpha(\Omega)$ ? In this note, we also consider this problem.

## 2 Function spaces

We first introduce Banach spaces of Morrey type and Besov type which are wider than  $L^\infty$ .

**DEFINITION.** (1)  $M_\beta(\Omega) = \{f \in L^1(\Omega); \|f\|_{M_\beta} < \infty\}$  is introduced by the norm

$$\|f\|_{M_\beta(\Omega)} := \sup_{x \in \mathbb{R}^n, 0 < t < 1} \frac{1}{|B(x, t)| \log^\beta(e + \frac{1}{t})} \int_{B(x, t)} E_0 |f(y)| dy$$

where  $E_0$  is the 0-extension operator from functions defined on  $\Omega$  to functions on  $\mathbb{R}^n$  and  $B(x, t) := \{y \in \mathbb{R}^n; |y - x| < t\}$ .

(2) (Modified Vishik's space). Let  $\psi \in \mathcal{S}(\mathbb{R}^n)$  be a spherical symmetric function with  $\hat{\psi}(\xi) = 1$  in  $B(0, 1)$  and  $\hat{\psi}(\xi) = 0$  in  $B(0, 2)^c$ .  $V_\beta(\mathbb{R}^n) = \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{V_\beta} < \infty\}$  is introduced by the norm

$$\|f\|_{V_\beta} := \sup_{N=1,2,\dots} \frac{\|\psi_N * f\|_\infty}{N^\beta}, \quad \text{where } \psi_N(x) := 2^{nN} \psi(2^N x).$$

We note that the space  $V_\beta$  is a modified version of spaces introduced by Vishik[14]. We also note that  $M_1(\Omega) \supset bmo(\Omega) \supset L^\infty(\Omega)$  and  $V_1(\mathbb{R}^n) \supset B_{\infty,\infty}^0(\mathbb{R}^n) \supset bmo(\mathbb{R}^n) \supset L^\infty(\mathbb{R}^n)$ .

Let  $C_{0,\sigma}^\infty(\Omega) = C_{0,\sigma}^\infty$  denote the set of all  $C^\infty$ -real vector fields  $\phi = (\phi^1, \dots, \phi^n)$  with compact support in  $\Omega$  such that  $\text{div } \phi = 0$ . Then  $L_\sigma^r$ ,  $1 < r < \infty$ , is the closure of  $C_{0,\sigma}^\infty$  with respect to the  $L^r$ -norm  $\|\cdot\|_r$ . Concerning Sobolev spaces we use the notations  $W^{k,p}(\Omega)$  and  $W_0^{k,p}(\Omega)$ ,  $k \in \mathbb{N}$ ,  $1 \leq p \leq \infty$ . Note that very often we will simply write  $L^r$  and  $W^{k,p}$  instead of  $L^r(\Omega)$  and  $W^{k,p}(\Omega)$ , respectively. The symbol  $(\cdot, \cdot)$  denotes the  $L^2$ -inner product and the duality pairing between  $L^p$  and  $L^{p'}$ , where  $1/p + 1/p' = 1$ .

Let us recall the Helmholtz decomposition:  $L^r(\Omega) = L_\sigma^r \oplus G_r$  ( $1 < r < \infty$ ), where  $G_r = \{\nabla p \in L^r; p \in L_{loc}^r(\bar{\Omega})\}$ ,  $P_r$  denotes the projection operator from  $L^r$  onto  $L_\sigma^r$  along  $G_r$ . The Stokes operator  $A_r$  on  $L_\sigma^r$  is defined by  $A_r = -P_r \Delta$  with domain  $D(A_r) = W^{2,r} \cap W_0^{1,r} \cap L_\sigma^r$ . Since  $P_r u = P_q u$  for all  $u \in L^r \cap L^q$  ( $1 < r, q < \infty$ ) and since  $A_r u = A_q u$  for all  $u \in D(A_r) \cap D(A_q)$ , for simplicity, we shall abbreviate  $P_r u, P_q u$  as  $Pu$  for  $u \in L^r \cap L^q$  and  $A_r u, A_q u$  as  $Au$  for  $u \in D(A_r) \cap D(A_q)$ , respectively.

In this paper, we denote by  $C$  various constants.

### 3 Main Theorems

Now our results read as follows:

**Theorem 1.** *Let  $\Omega(\subset \mathbb{R}^n)$  be a bounded domain with  $\partial\Omega \in C^\infty$ .*

(1) *For any  $\alpha \in (0, 1)$  and  $\beta > 0$ , there exists a constant  $C(\Omega, \alpha, \beta) > 0$  such that*

$$(3.1) \quad \|f\|_{L^\infty(\Omega)} \leq C \left( 1 + \|f\|_{M_\beta(\Omega)} \log^\beta (e + \|f\|_{\dot{C}^\alpha(\Omega)}) \right) \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap M_\beta(\Omega).$$

(2) *Let  $\beta > 0$  and  $X$  be a normed space. Assume that  $X$  satisfies the following conditions:*

$$(A) \quad \left\{ \begin{array}{ll} (i) & L^\infty \hookrightarrow X \subset L^1(\Omega), \\ (ii) & X \text{ is a translation invariant space, i.e.,} \\ & \|f(\cdot - y)\|_X = \|f\|_X \text{ if } f \text{ and } y \in \mathbb{R}^n \text{ satisfy } \text{supp } f(\cdot - y), \text{supp } f \subset \bar{\Omega}, \\ (iii) & \|f\|_X \leq \|g\|_X \text{ if } |f(x)| \leq |g(x)| \text{ a.e. } x \in \Omega, \\ (iv) & \text{there exist constants } \alpha \in (0, 1) \text{ and } C > 0 \text{ such that} \\ & \|f\|_{L^\infty(\Omega)} \leq C \left( 1 + \|f\|_X \log^\beta (e + \|f\|_{\dot{C}^\alpha(\Omega)}) \right) \text{ for all } f \in \dot{C}^\alpha(\Omega) \cap X. \end{array} \right.$$

*Then,  $X$  is continuously imbedded in  $M_\beta(\Omega)$ .*

**Remarks.** Since  $M_\beta(\Omega)$  satisfies (A), Theorem 1 implies that  $M_\beta$  is the largest normed space that satisfies conditions (A).

When we do not assume (A)(iii), there is a normed space wider than  $M_\beta$  if  $\Omega = \mathbb{R}^n$  as below.

**Theorem 2.** (1) For any  $\alpha \in (0, 1)$  and  $\beta > 0$ , there exists a constant  $C(\alpha, \beta) > 0$  such that

$$(3.2) \quad \|f\|_{L^\infty(\mathbb{R}^n)} \leq C \left( 1 + \|f\|_{V_\beta(\mathbb{R}^n)} \log^\beta (e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}) \right) \text{ for all } f \in \dot{C}^\alpha(\mathbb{R}^n) \cap V_\beta.$$

(2) Let  $\beta > 0$  and  $X$  be a normed space. Assume that  $X$  satisfies the following conditions:

$$(B) \quad \left\{ \begin{array}{ll} (i) & X \hookrightarrow \mathcal{S}'(\mathbb{R}^n), \\ (ii) & X \text{ is a translation invariant space, i.e.,} \\ & \|f(\cdot - y)\|_X = \|f\|_X \text{ for all } y \in \mathbb{R}^n, \\ (iii) & \|\rho * f\|_X \leq \|\rho\|_{L^1(\mathbb{R}^n)} \|f\|_X \text{ for all } \rho \in \mathcal{S}' \\ (iv) & \text{there exist constants } \alpha \in (0, 1) \text{ and } C > 0 \text{ such that} \\ & \|f\|_{L^\infty(\mathbb{R}^n)} \leq C \left( 1 + \|f\|_X \log^\beta (e + \|f\|_{\dot{C}^\alpha(\mathbb{R}^n)}) \right) \text{ for all } f \in BC^\infty(\mathbb{R}^n). \end{array} \right.$$

Then,  $X$  is continuously imbedded in  $V_\beta$ .

**Remark.** (i) Since  $V_\beta(\mathbb{R}^n)$  satisfies (B), Theorem 2 implies that  $V_\beta(\mathbb{R}^n)$  is the largest normed space that satisfies conditions (B).

(ii) Since  $E_0 M_\beta(\Omega) \subset V_\beta(\mathbb{R}^n)$ ,  $V_\beta(\mathbb{R}^n)$  can be regarded as a space wider than  $M_\beta$ .

**Theorem 3.** Let  $\Omega(\subset \mathbb{R}^3)$  be a bounded domain with  $\partial\Omega \in C^\infty$ ,  $p \geq 3$  and  $u$  be a solution to (N-S) on  $(0, T)$  in the class

$$S_p(0, T) := C([0, T]; L_\sigma^p) \cap C^1((0, T); L_\sigma^p) \cap C((0, T); W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)).$$

Assume that  $T < \infty$  and  $T$  is maximal, i.e.,  $u$  cannot be continued to the solution of (N-S) in the class  $S_p(0, T')$  for any  $T' > T$ . Then,

$$\int_s^t \|\omega(\tau)\|_{M_1(\Omega)} d\tau \nearrow \infty \quad \text{as } t \nearrow T \quad \text{for any } s \in (0, T).$$

**Remark.** When  $\Omega = \mathbb{R}^3$ , under the same assumptions, we have  $\int_s^t \|\omega(\tau)\|_{V_1(\mathbb{R}^3)} d\tau \nearrow \infty$  as  $t \nearrow T$  for any  $s \in (0, T)$ .

## 4 Proof of main theorems

**Proof of Theorem 1(1).** For the proof of Theorem 1(2), see our paper[10]. Here, we prove only Theorem 1(1). To this end, we use an argument given in Ozawa [12]. See also [5] and [11]. For the sake of simplicity, we assume  $n = 3$ . Since  $\partial\Omega \in C^\infty$ ,  $\partial\Omega$  satisfies the interior cone condition. Namely there are  $\delta \in (0, 1)$  and  $\theta \in (\pi/2, \pi)$  depending only on  $\Omega$  with the following property: For any point  $x \in \Omega$ , there exists a spherical sector  $C_\delta^\theta(x) = \{x + \xi \in \mathbb{R}^3; 0 < |\xi| \leq \delta, -|\xi| \leq \kappa(x) \cdot \xi \leq |\xi| \cos \theta\}$  having a vertex at  $x$  such that  $C_\delta^\theta(x) \subset \Omega$ , where  $\kappa(x)$  is an appropriate unit vector from  $x$ . We note that for each  $x \in \Omega$ ,  $C_\delta^\theta(x)$  is congruent to  $C_\delta^\theta \equiv \{\xi \in \mathbb{R}^3; 0 < |\xi| \leq \delta, -|\xi| \leq \xi_3 \leq |\xi| \cos \theta\}$ . In particular, for any boundary point  $x \in \partial\Omega$ ,  $C_\delta^\theta(x)$  can be expressed as  $C_\delta^\theta(x) \equiv \{x + \xi \in \mathbb{R}^3; 0 < |\xi| \leq \delta, -|\xi| \leq \xi \cdot \nu(x) \leq |\xi| \cos \theta\}$ , where  $\nu(x)$  denotes the unit outward normal at  $x$ .

Let  $0 < t \leq \delta$  and  $C_t^\theta(x) := C_\delta^\theta(x) \cap \overline{B(x, t)}$ . For any fixed  $x \in \Omega$  and  $y \in C_t^\theta(x) \subset \Omega$ ,

$$|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq \|f\|_{\dot{C}^\alpha} |x - y|^\alpha + |f(y)| \leq \|f\|_{\dot{C}^\alpha} t^\alpha + |f(y)|$$

Integrating both sides of above inequality with respect to  $y$  over  $C_t^\theta(x)$ ,

$$\begin{aligned} |f(x)| |C_t^\theta(x)| &\leq t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} |C_t^\theta(x)| + \int_{y \in C_t^\theta(x)} |f(y)| dy \\ (4.1) \quad &\leq t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} |C_t^\theta(x)| + \int_{y \in B(x, t) \cap \Omega} |f(y)| dy \\ &\leq t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} |C_t^\theta(x)| + |B(x, t)| \log^\beta\left(\frac{1}{t} + e\right) \|f\|_{M_\beta(\Omega)} \end{aligned}$$

Since  $|B(x, t)|/|C_t^\theta(x)| (< \infty)$  is only depending on  $\theta$ , we have

$$(4.2) \quad |f(x)| \leq t^\alpha \|f\|_{\dot{C}^\alpha(\Omega)} + C \log^\beta\left(\frac{1}{t} + e\right) \|f\|_{M_\beta(\Omega)}$$

for all  $0 < t < \delta$ .

Then we optimize  $t$  by letting  $t = (1/\|f\|_{\dot{C}^\alpha(\Omega)})^{1/\alpha}$  if  $\|f\|_{\dot{C}^\alpha(\Omega)} \geq \delta^{-\alpha}$  and letting  $t = \delta$  if  $\|f\|_{\dot{C}^\alpha(\Omega)} \leq \delta^{-\alpha}$  to obtain (3.1). □

**Proof of Theorem 2(1).** Here, we prove Theorem 2(1). For the proof of Theorem 2(2), see our paper[10]. We first recall the Littlewood-Paley decomposition. Let  $\psi$  be the function given in Definition (2) and let  $\phi_j \in \mathcal{S}$  be the functions defined by

$$\hat{\phi}(\xi) := \hat{\psi}(\xi) - \hat{\psi}(2\xi) \text{ and } \hat{\phi}_j(\xi) := \hat{\phi}(\xi/2^j) \text{ for } \xi \in \mathbb{R}^3.$$

Then,  $\text{supp } \hat{\phi}_j \subset \{2^{j-1} \leq |\xi| \leq 2^{j+1}\}$  and

$$(4.3) \quad 1 = \hat{\psi}(\xi/2^N) + \sum_{j=N+1}^{\infty} \hat{\phi}(\xi/2^j) = \hat{\psi}_N(\xi) + \sum_{j=N+1}^{\infty} \hat{\phi}_j(\xi) \text{ for } \xi \in \mathbb{R}^3, N = 1, 2, \dots$$

Using (4.3), we decompose  $f$  into two parts such as

$$(4.4) \quad f(x) = \psi_N * f(x) + \sum_{j=N+1}^{\infty} \phi_j * f(x).$$

We have by Definition (2)

$$(4.5) \quad \|\psi_N * f\|_{\infty} \leq N^{\beta} \|f\|_{V_{\beta}}.$$

Since  $\dot{B}_{\infty, \infty}^{\alpha} = \dot{C}^{\alpha}$  for  $0 < \alpha < 1$ , we have

$$(4.6) \quad \begin{aligned} \sum_{j=N+1}^{\infty} \|\phi_j * f\|_{\infty} &\leq C \sum_{j=N+1}^{\infty} 2^{\alpha j} \|\phi_j * f\|_{\infty} 2^{-\alpha j} \\ &\leq C \|f\|_{\dot{B}_{\infty, \infty}^{\alpha}} \sum_{j=N+1}^{\infty} 2^{-\alpha j} \leq C \|f\|_{\dot{C}^{\alpha}} 2^{-\alpha N}. \end{aligned}$$

Gathering (4.5) and (4.6) with (4.4), we obtain

$$(4.7) \quad \|f\|_{\infty} \leq C(2^{-\alpha N} \|f\|_{\dot{C}^{\alpha}} + N^{\beta} \|f\|_{V_{\beta}}).$$

Now we take  $N = \left\lceil \frac{\log(\|f\|_{\dot{C}^{\alpha}} + e)}{\alpha \log 2} \right\rceil + 1$ , where  $\lceil \cdot \rceil$  denotes Gauss symbol. Then we have the desired estimate (3.2)  $\square$

*Proof of Theorem 3.* For the sake of simplicity, we prove Theorem 3 only in the case  $p > 3$ . Since  $u \in C((0, T); D(A_p))$ , without loss of generality, we may assume that  $u_0 \in D(A_p)$ . Since  $P(u \cdot \nabla u) = P(\omega \times u)$ ,  $u$  satisfies the following integral equation:

$$(I.E.) \quad u(t) = e^{-tA} u_0 - \int_0^t e^{-(t-s)A} P(\omega \times u)(s) ds$$

for all  $0 < t < T$ . Since the local existence time  $T_*$  of  $L^p$ -strong solutions can be estimated from below as

$$T_* > C(p, \Omega) / \|u_0\|_p^{2p/(p-3)},$$

we observe that if we assume

$$(4.8) \quad \sup_{0 < t < T} \|u(t)\|_p < \infty,$$

then  $u$  can be continued to the solution of (N-S) in the class  $S_p(0, T')$  for some  $T' > T$ . Hence, in order to prove Theorem 3, it suffices to show that if

$$(4.9) \quad \int_0^T \|\omega(t)\|_{M_1} dt < \infty,$$

then (4.8) holds. From now on we assume (4.9). By (I.E.) we have

$$\|u(t)\|_p \leq C\|u_0\|_p + C \int_0^t \|\omega(s)\|_\infty \|u(s)\|_p ds,$$

which yields

$$(4.10) \quad \sup_{0 < s < t} \|u(s)\|_p \leq C\|u_0\|_p \exp \left( C \int_0^t \|\omega(\tau)\|_\infty d\tau \right)$$

for all  $0 < t \leq T$ . Therefore, in order to show (4.8), it suffices to show

$$(4.11) \quad \int_0^T \|\omega(\tau)\|_\infty d\tau < \infty.$$

Letting  $\alpha > 0$  and substituting  $f = \frac{\omega(s)}{\epsilon \|\omega(s)\|_{\dot{C}^\alpha}}$  into the Brezis-Gallouet-Wainger type inequality (3.1) with  $\beta = 1$ , we obtain

$$(4.12) \quad \begin{aligned} \|\omega(s)\|_\infty &\leq C \left( \epsilon \|\omega(s)\|_{\dot{C}^\alpha} + \log \left( e + \frac{1}{\epsilon} \right) \|\omega(s)\|_{M_1} \right) \\ &\leq C \left( \epsilon \|u(s)\|_{C^{1+\alpha}} + \log \left( e + \frac{1}{\epsilon} \right) \|\omega(s)\|_{M_1} \right) \end{aligned}$$

for all  $\epsilon > 0$ , where  $C$  is a constant independent of  $s$  and  $\epsilon$ . Let  $0 < \alpha < 1 - 3/p$  and

$$\begin{aligned} h(t) &:= \sup_{0 < \tau < t} \|u(\tau)\|_p, \\ g(t) &:= \int_0^t \|\omega(\tau)\|_\infty d\tau \end{aligned}$$

for  $0 < t < T$ . Then, from (4.12), for any positive function  $\epsilon(s)$  on  $(0, T)$  we see that

$$(4.13) \quad \begin{aligned} g(t) &\leq C \int_0^t \|u(s)\|_{C^{1+\alpha}} \epsilon(s) ds + C \int_0^t \log \left( e + \frac{1}{\epsilon(s)} \right) \|\omega(s)\|_{M_1} ds \\ &=: I_1(t) + I_2(t). \end{aligned}$$

Since

$$\|e^{-tA} f\|_{C^{1+\alpha}} \leq C(1 + t^{-\frac{1+\alpha}{2} - \frac{3}{2p}}) \|f\|_p,$$



from (I.E.) we obtain

$$\begin{aligned}
 \|u(s)\|_{C^{1+\alpha}} &\leq C\|u_0\|_{D(A_p)} + C \int_0^s (1 + (s - \tau)^{-\frac{1+\alpha}{2} - \frac{3}{2p}}) \|\omega \times u(\tau)\|_p d\tau \\
 &\leq C\|u_0\|_{D(A_p)} + Ch(s) \int_0^s (1 + (s - \tau)^{-\frac{1+\alpha}{2} - \frac{3}{2p}}) \|\omega(\tau)\|_\infty d\tau.
 \end{aligned}
 \tag{4.14}$$

Hence, for  $0 < t < T$  we have

$$\begin{aligned}
 I_1(t) &\leq C\|u_0\|_{D(A_p)} T \sup_{0 < s < T} \epsilon(s) \\
 &\quad + C \int_0^t h(s) \epsilon(s) \int_0^s (1 + (s - \tau)^{-\frac{1+\alpha}{2} - \frac{3}{2p}}) \|\omega(\tau)\|_\infty d\tau ds
 \end{aligned}
 \tag{4.15}$$

We now choose  $\epsilon(s)$  such that

$$\epsilon(s) := \frac{\delta}{Ch(s) + 1},$$

where  $\delta > 0$  is a sufficiently small constant. Then, by Fubini's Theorem we have

$$I_1(t) \leq CT\delta\|u_0\|_{D(A_p)} + C_0(T)\delta \int_0^t \|\omega(\tau)\|_\infty d\tau.$$

Since (4.10) yields

$$\log(e + \frac{1}{\epsilon(s)}) = \log(e + \frac{Ch(s) + 1}{\delta}) \leq C(\delta) (\log(e + \|u_0\|_p) + g(s)),$$

we have

$$I_2(t) \leq C(\delta) \int_0^t \|\omega(s)\|_{M_1} (\log(e + \|u_0\|_p) + g(s)) ds$$

Gathering (4.16) and (4.17) with (4.13) we obtain

$$g(t) \leq CT\delta\|u_0\|_{D(A_p)} + C_0(T)\delta g(t) + C(\delta) \int_0^t \|\omega(s)\|_{M_1} (\log(e + \|u_0\|_p) + g(s)) ds.$$

Therefore, letting  $\delta = 1/(2C_0(T))$ , by the Gronwall lemma, we have

$$g(t) \leq C(\|u_0\|_{D(A_p)}, T, \|u_0\|_p) \exp \left( C(T) \int_0^T \|\omega(s)\|_{M_1} ds \right)$$

for all  $0 < t < T$ , which implies the desired estimate (4.11).

□

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